CONVERGENCE ABSCISSAS FOR DIRICHLET SERIES WITH MULTIPLICATIVE COEFFICIENTS

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ABSTRACT. This note deals with the relationship between the abscissas of simple, uniform and absolute convergence for the Dirichlet series $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, when the coefficients a_n are either multiplicative or completely multiplicative.

Consider the ordinary Dirichlet series

$$f(s) = \sum_{n=1}^{\infty} a_n n^{-s}, \qquad s = \sigma + it.$$

A basic fact is that Dirichlet series converge in half-planes, just as power series converge in discs. However, Dirichlet series can have different types of convergence in distinct half-planes. It was H. Bohr [4, 6] who first studied the relationship between the following three convergence abscissas:

$$\sigma_c(f) = \inf\left\{\sigma : \sum_{n=1}^{\infty} a_n n^{-\sigma} \text{ converges}\right\} \qquad \text{(Simple)},$$

$$\sigma_b(f) = \inf\left\{\sigma : \sum_{n=1}^{\infty} a_n n^{-\sigma-it} \text{ converges uniformly for } t \in \mathbb{R}\right\} \qquad \text{(Uniform)},$$

$$\sigma_a(f) = \inf\left\{\sigma : \sum_{n=1}^{\infty} |a_n| n^{-\sigma} \text{ converges}\right\} \qquad \text{(Absolute)}.$$

Clearly $\sigma_c \leq \sigma_b \leq \sigma_a$, and it is easy to deduce that $\sigma_a(f) - \sigma_c(f) \leq 1$. Under the assumption that the Dirichlet series f does not converge at $s = 0$, the Cauchy–Hadamard type formulas for these abscissas are:

$$\sigma_c(f) = \limsup_{x\to\infty} \frac{1}{\log x} \log\left|\sum_{n\leq x} a_n\right|,$$

$$\sigma_b(f) = \limsup_{x\to\infty} \frac{1}{\log x} \log\left(\sup_{t\in\mathbb{R}} \left|\sum_{n\leq x} a_n n^{-it}\right|\right),$$

$$\sigma_a(f) = \limsup_{x\to\infty} \frac{1}{\log x} \log\left(\sum_{n\leq x} |a_n|\right).$$

By choosing $a_n = \pm 1$ in a suitable manner, it is now easy to construct a Dirichlet series with $\sigma_a - \sigma_c = \alpha$, for any $\alpha \in [0,1]$. Moreover, the Cauchy–Schwarz inequality can be applied to show that $\sigma_a - \sigma_b \leq 1/2$. The fact that there are Dirichlet series with $\sigma_a - \sigma_b = \beta$ for any $\beta \in [0, 1/2]$

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is a result due to Bohnenblust–Hille [3]. See [2] for an excellent exposition of these results, containing clear proofs using modern techniques.

The inequality used in [3] to obtain this result was recently substantially improved [1, 11], and the improved version can be used to get a precise qualitative version of the optimality of $\beta = 1/2$ in view of the Cauchy–Hadamard formulas given above (see [10]).

It is interesting to consider the difference between these abscissas when the coefficients have some added multiplicative structure (recall that a_n is *multiplicative* if $a_{mn} = a_m a_n$ whenever $\gcd(m,n) = 1$ and is *completely multiplicative* if this relationship persists for any choice of m and n). For example, the Riemann hypothesis is equivalent to $\sigma_a - \sigma_c = 1/2$ for the series

$$1/\zeta(s) = \sum_{n=1}^{\infty} \mu(n) n^{-s} = \prod_p \left(1 - p^{-s}\right),$$

where $\mu(n)$ is the Möbius function, which of course is multiplicative.

Lévy [16] argued that any random model of the Möbius function should take into account the multiplicative nature of $\mu(n)$, and, following this, Wintner [17] showed that the Dirichlet series represented by the Euler product

$$\prod_p \left(1 + \varepsilon_p p^{-s}\right)$$

has $\sigma_c = 1/2$ almost always, and concluded that "the Riemann hypothesis is almost always true". Here ε_p denotes the Rademacher random variables which assumes the values ± 1 with equal probability.

Motivated by this result regarding "typical" behavior, we will investigate the possible values for $\sigma_a(f) - \sigma_c(f)$ and $\sigma_a(f) - \sigma_b(f)$, when the coefficients of the Dirichlet series f are either multiplicative or completely multiplicative. For the first quantity, we have the following.

Theorem 1. *There exists a Dirichlet series f with completely multiplicative coefficients such that $\sigma_a(f) - \sigma_c(f) = \alpha$ for any $\alpha \in [0,1]$.*

Proof. The cases $\alpha = 0$ and $\alpha = 1$ follow from considering the Riemann zeta function and the Dirichlet L-function of a non-principal character, respectively.

For $0 < \alpha < 1$, consider

$$g_\alpha(s) = \left(1 - 3^{1-\alpha-s}\right)^{-1} = \sum_{k=0}^{\infty} 3^{(1-\alpha)k} 3^{-ks}.$$

We now let χ denote the non-principal character of modulus 3 and we consider the Dirichlet series given by the product

$$f(s) = g_\alpha(s) L(s,\chi).$$

Clearly, $f(s)$ has completely multiplicative coefficients, since $\chi(3) = 0$ and since $g_\alpha(s)$ is a geometric series. The latter fact also implies that $\sigma_c(g_\alpha) = \sigma_a(g_\alpha) = 1-\alpha$, and for the L-function of a non-principal character we have $\sigma_c = 0$ and $\sigma_a = 1$. Now, the product of a conditionally convergent series and an absolutely convergent series is conditionally convergent, so we have $\sigma_c(f) \le 1-\alpha$. This cannot be improved, since $f(1-\alpha)$ does not convergence (an infinite number of the terms have modulus 1), so $\sigma_c(f) = 1-\alpha$.

The product of two absolutely convergent series is absolutely convergent, so $\sigma_a(f) \le 1$. We let $|f|(s)$ denote the Dirichlet series where we have replaced the coefficients by their absolute values. We see that $|f|(1)$ diverges since $L(1,|\chi|)$ diverges, the coefficients of g_α are positive, and $g_\alpha(1) \neq 0$. In conclusion, we have $\sigma_a(f) - \sigma_c(f) = 1 - (1-\alpha) = \alpha$. ☐

Of course, $g_\alpha(s)$ can be replaced by any power series in 3^{-s} with non-negative coefficients and $\sigma_a = 1-\alpha$ to obtain an example which is multiplicative, but not completely multiplicative.

Our next result can be considered as an example of the following scheme: A *contractive* function theoretic result concerning power series, can possibly be applied *multiplicatively* to obtain a similar result for ordinary Dirichlet series. A recent example of this type of result is [9, Thm. 2]. See also the proof of the main theorem in [14].

Theorem 2. *Suppose that the Dirichlet series f has multiplicative coefficients. Then $\sigma_a = \sigma_b$.*

It was H. Bohr who realized the connection between Dirichlet series and function theory in polydiscs [5], through the correspondence $p_j^{-s} \leftrightarrow z_j$. Inspecting the prime factorization $n = \prod_j p_j^{\alpha_j}$, we associate to the integer n the multi-index $\alpha(n) = (\alpha_1, \alpha_2, \ldots)$. The *Bohr lift* of the Dirichlet series $f(s) = \sum_{n\geq 1} a_n n^{-s}$ is the power series

$$\mathscr{B}f(z) = \sum_{n=1}^{\infty} a_n z^{\alpha(n)}.$$

Using Kronecker's theorem [12, Ch. 13] (see also [13, Sec. 2.2]), we may conclude that

$$\|f\|_\infty := \sup_{\sigma>0} |f(s)| = \sup_{z\in\mathbb{D}^\infty\cap c_0} |\mathscr{B}f(z)|.$$

Now, let us suppose that f has multiplicative coefficients. We may then factor

$$f(s) = \prod_j \left(1 + \sum_{k=1}^{\infty} a_{p_j^k} p^{-ks}\right) = \prod_j f_j(s),$$

at least for $\sigma > \sigma_a$. In particular, since each prime only appears in one factor, we also obtain

$$\|f\|_\infty = \sup_{z\in\mathbb{D}^\infty\cap c_0} |\mathscr{B}f(z)| = \prod_j \sup_{z_j\in\mathbb{D}} |\mathscr{B}f_j(z_j)| = \prod_j \|f_j\|_\infty.$$

To complete the proof of Theorem 2, we will require the following.

Lemma. *Let $F(z) = \sum_{m\geq 0} b_m z^m$ and suppose that $\sup_{z\in\mathbb{D}} |F(z)| < \infty$. Let $0 \leq r < 1$. Then*

$$\sum_{m=0}^{\infty} |b_m| r^m \leq C(r) \sup_{z\in\mathbb{D}} |F(z)|,$$

where

$$C(r) = \begin{cases} 1, & 0 \leq r \leq 1/3, \\ 1/\sqrt{1-r^2}, & 1/3 < r < 1. \end{cases}$$

Proof. The first estimate is Bohr's inequality [7], the second follows from the Cauchy–Schwarz inequality, Parseval's formula and the maximum modulus principle. □

The contractive function theoretic result for power series mentioned earlier is that $C(r) = 1$ when $0 \leq r \leq 1/3$. It should also be pointed out that the values $C(r)$ prescribed above are not optimal when $r > 1/3$, and that precise estimates in this range can be found in [8].

Proof of Theorem 2. Let the coefficients of $f(s) = \sum_{n\geq 1} a_n n^{-s}$ be multiplicative, and fix $\varepsilon > 0$. Since uniform convergence implies boundedness, we may (after a horizontal translation) assume that $\sigma_b(f) = -\varepsilon$ so that $\|f\|_\infty < \infty$. We then want to prove that under this assumption we have

$$\sum_{n=1}^{\infty} |a_n| n^{-\varepsilon} < \infty,$$

so that $\sigma_a(f) \le \varepsilon$, and hence $\sigma_a(f) - \sigma_b(f) \le 2\varepsilon$. Since $\varepsilon > 0$ is arbitrary, $\sigma_a(f) = \sigma_b(f)$. By the discussion preceding it and the lemma, we obtain

$$\sum_{n=1}^{\infty} |a_n| n^{-\varepsilon} = \prod_p \left(1 + \sum_{k=1}^{\infty} |a_{p^k}| p^{-k\varepsilon}\right) \le \left(\prod_{p^\varepsilon < 3} \frac{\|f_p\|_\infty}{\sqrt{1-p^{-2\epsilon}}}\right)\left(\prod_{3 \le p^\varepsilon < \infty} 1 \cdot \|f_p\|_\infty\right)$$
$$= \left(\prod_{p^\varepsilon < 3} \frac{1}{\sqrt{1-p^{-2\epsilon}}}\right)\left(\prod_p \|f_p\|_\infty\right) = \left(\prod_{p^\varepsilon < 3} \frac{1}{\sqrt{1-p^{-2\epsilon}}}\right) \|f\|_\infty < \infty. \qquad \square$$

Theorem 2 allows us to provide a strengthening of a result of Bohr in the case of Dirichlet series with multiplicative coefficients.

Corollary. *Let $f(s) = \sum_{n \ge 1} a_n n^{-s}$ have multiplicative coefficients and suppose that f is somewhere convergent. If f has a bounded analytic continuation to $\sigma \ge \sigma_0 + \varepsilon$, for every $\varepsilon > 0$, then $\sigma_a(f) = \sigma_0$.*

Proof. Bohr's theorem states that $\sigma_b(f) = \sigma_0$ without any assumptions on the coefficients of f. By Theorem 2, we have $\sigma_a(f) = \sigma_b(f) = \sigma_0$. $\square$

Note added in proof

In a recent paper [15], J. Kaczorowski and A. Perelli have independently proven Theorem 2 under the additional assumption that the Dirichlet series belongs to the Selberg class. Their methods are slightly different and do not involve analysis on the polydisc.

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